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ALL INDIA TEST SERIES CSE-2023

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5. **DATE:-** 23-07-23

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Q.NO	MARKS
1.	32½
2.	32½
3.	24½
4.	19½
5.	30
6.	
7.	
8.	

TOTAL MARKS	139/250

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Q1(a)

Scattering is deflection of a particle from its original path.

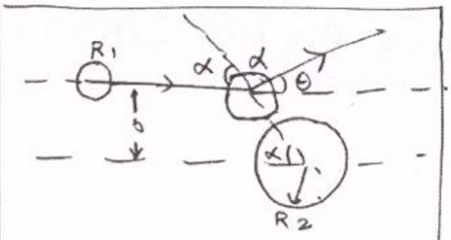

Fig 1. Scattering

> Let the sphere R_1 approach at height b above centre.

> $\angle\theta$ = scattering angle.

Now as sphere R_2 is massive, $|p_i| = |p_f|$

$\Rightarrow \angle$ b/w p_i & $p_f = 2\alpha = \pi - \theta$.

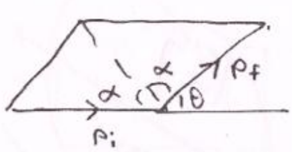

Fig 2. Momentum change

From figure, $b = (R_1+R_2)\sin\alpha$

$$\Rightarrow b = (R_1+R_2)\sin\left(\frac{\pi-\theta}{2}\right)$$

$$\Rightarrow b = (R_1+R_2)\cos\theta/2$$

We know, scattering cross-section $(\sigma(\theta))$

$$= \frac{-b}{\sin\theta}\frac{db}{d\theta} \Rightarrow \sigma(\theta) = -\frac{(R_1+R_2)\cos\theta/2}{\sin\theta}\cdot\frac{(R_1+R_2)-\sin\theta/2}{2}$$

$$\Rightarrow \boxed{\sigma(\theta) = \frac{(R_1+R_2)^2}{4}}$$

Total cross section $= \int_0^{\pi} \sigma(\theta)\sin\theta\, 2\pi\, d\theta$

$$\Rightarrow \sigma = \text{total cross-section} = \boxed{\pi(R_1+R_2)^2}.$$

Scattering cross-section gives the no. of particles scattered per unit area · time per unit incident intensity

$6\frac{1}{2}$ / 10

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Q.1 (b)

Central force is a motion in which force is solely dependent on position of particle.

> For central force, $F_{(r)} = \frac{-J^2u^2}{m}\left[\frac{d^2u}{d\theta^2} + u\right]$

where J = angular momentum ; $u = 1/r$.

$\Rightarrow u = \frac{1}{a}\frac{1}{1+\cos\theta} \Rightarrow \frac{du}{d\theta} = \frac{1}{a}\frac{-1}{(1+\cos\theta)^2}(-\sin\theta)$

$\Rightarrow \frac{du}{d\theta} = \frac{1}{a}\frac{\sin\theta}{(1+\cos\theta)^2} \Rightarrow \frac{d^2u}{d\theta^2} = \frac{1}{a}\left[\frac{\cos\theta(1+\cos\theta)^2 - \sin\theta\, 2(1+\cos\theta)(-\sin\theta)}{(1+\cos\theta)^4}\right]$

$\Rightarrow \frac{d^2u}{d\theta^2} = \frac{1}{a}\left[\frac{\cos\theta(1+\cos\theta)^2 + 2\sin^2\theta(1+\cos\theta)}{(1+\cos\theta)^4}\right]$ $(1-\cos^2\theta)$

~~$\Rightarrow \frac{d^2u}{d\theta^2} = \frac{1}{a}\left[\frac{\cos\theta + \cos^3\theta + 2\cos^2\theta + 2\sin^2\theta + 2\sin^2\theta\cos\theta}{(1+\cos\theta)^4}\right]$~~

$\Rightarrow \frac{d^2u}{d\theta^2} = \frac{1}{a}\left[\frac{-\cos^3\theta + 3\cos\theta + 2}{(1+\cos\theta)^4}\right] \Rightarrow \frac{d^2u}{d\theta^2} + u =$

$\frac{1}{a}\left[\frac{-\cos^3\theta + 3\cos\theta + 2}{(1+\cos\theta)^4} + \frac{(1+\cos\theta)^3}{(1+\cos\theta)^4}\right] = \frac{3}{a}\frac{1}{(1+\cos\theta)^2}$

$\Rightarrow F_{(r)} = -\frac{J^2u^2}{m}\cdot 3au^2 = \boxed{-\frac{3J^2a}{m}\left(\frac{1}{r^4}\right)}$

$\Rightarrow$ **Force is $\propto \frac{1}{r^4}$**

$6\frac{1}{2}$ / 10

Q1(c) Einstein's special theory of relativity gave rise to concepts of relativistic mass & energy.

s.t.

$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$. Given $m = 2m_0 \Rightarrow 1 - \frac{v^2}{c^2} = 1/4$

$\Rightarrow v = \sqrt{3}/2\, c$ — (1)

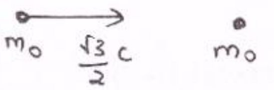

Fig 1. collision

Conserving relativistic energy,

$2m_0c^2 + m_0c^2 = mc^2$

$\Rightarrow \boxed{m = 3m_0}$ ∴ Mass of resulting particle = $3m_0$.

Conserving momentum,

$2m_0\left(\frac{\sqrt{3}c}{2}\right) = 3m_0(v') \Rightarrow \boxed{v' = \frac{c}{\sqrt{3}}}$

∴ Mass of resulting particle is $\boxed{3m_0}$ while velocity is $\boxed{c/\sqrt{3}}$

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Q1 (d)

In automobiles, piston produce rotation of axle which provides torque to the wheels

We know power $(P) = \tau\omega$ } τ = torque, ω = angular velocity

Given : 1800 rev/min $\Rightarrow \frac{1800 \times 2\pi}{60}$ rad/sec

$\Rightarrow \omega = 60\pi$ rad/sec.

$\Rightarrow P = 75 \times 10^3 = \tau \times 60\pi$

$\Rightarrow$ Torque $= 3.977 \times 10^2$ N-m

$\therefore$ Torque delivered by automobile engine is ~ 397.7 N-m

6½/10

Max. torque is used to determine max. stress & hence leads to appropriate material choice.

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Answer Questions in NOT MORE THAN the Word Limit specified for each in the Parenthesis.
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Q1(e). Earth's sudden contraction will change its radius ⟹ it will change its moment of inertia.

> Angular momentum will have to be conserved

$\Rightarrow I_1 \omega_1 = I_2 \omega_2$ } Assuming earth perfect sphere, $I = 2/5 MR^2$.

$\Rightarrow \frac{2}{5} M R^2 \cdot \omega_1 = \frac{2}{5} M \left(\frac{R}{4}\right)^2 \cdot \omega_2$

$\Rightarrow \omega_2 = 16\omega_1$ ⟹ New angular velocity will be 16x current ⟹ New day length would be $\frac{1}{16}$ of current

⟹ New day length = $\frac{24}{16}$ = 1.5hrs.

Assuming mass, density remains constant.

Such high angular velocity will affect gravity significantly disturbing mountains, atmosphere, etc.

6½

Q.2 (a)

Particle in rotating frame of references experiences various ~~very~~ pseudo forces owing to relative motion between frame & particle.

We know for a rotating frame,

$$\left.\frac{d(\ )}{dt}\right]_{\text{space set}} = \left.\frac{d(\ )}{dt} + \vec{\omega}\times(\ )\right]_{\text{Body set}}$$ where body rotating with ω.

$$\Rightarrow \frac{d}{dt}(r) = \frac{d}{dt}(r') + \vec{\omega}\times(r)$$

$$\Rightarrow V = v' + \vec{\omega}\times\vec{r} \quad \text{— (1)}$$

Also $$\frac{d}{dt}(v) = \frac{d}{dt}(v' + \vec{\omega}\times\vec{r}) + \vec{\omega}\times(\vec{v}' + \vec{\omega}\times\vec{r})$$

$$\Rightarrow a = \frac{dv'}{dt} + \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d\vec{r}}{dt} + \vec{\omega}\times\vec{v}' + \vec{\omega}\times\vec{\omega}\times\vec{r}$$

$$\Rightarrow a = a' + \frac{d\vec{\omega}}{dt}\times\vec{r} + 2(\vec{\omega}\times\vec{v}') + \vec{\omega}\times\vec{\omega}\times\vec{r}$$

$$\Rightarrow ma' = \boxed{m\frac{d^2x'}{dt^2} = a - \underbrace{2(\vec{\omega}\times\vec{v}')}_{a_{co}} - \underbrace{\frac{d\vec{\omega}}{dt}\times r}_{a_E} - \underbrace{\vec{\omega}\times\vec{\omega}\times\vec{r}}_{a_{CE}}}$$

Where a_{co} = Coriolis force

a_E = Euler's force — directed tangentially

a_{CE} = Centrifugal force — directed radially outwards.

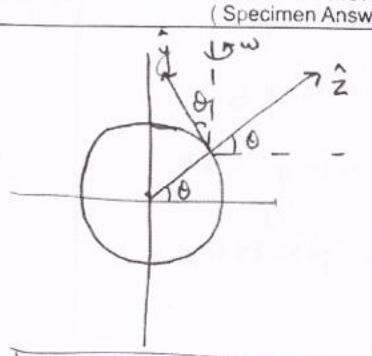

Fig 1. Directions assumed

$\equiv$

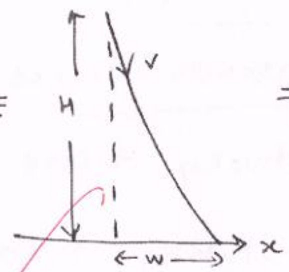

Fig 2. Falling particle

$\Rightarrow$ $\omega = \omega\cos\theta\,\hat{j} + \omega\sin\theta\,\hat{k}$

$V = -v\hat{k}$.

Here, deflection would be due to Coriolis force acting on particle.

Now $a_c = -2\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \omega\cos\theta & \omega\sin\theta \\ 0 & 0 & -v \end{vmatrix} = 2v\omega\cos\theta\,\hat{i}$ i.e. towards east side.

Now $v = gt \Rightarrow \frac{d^2x}{dt^2} = (2\omega\cos\theta)\,gt$

$\Rightarrow \frac{dx}{dt} = (2\omega\cos\theta)\,g\frac{t^2}{2} + c_1$ } At $t=0$, $dx/dt = 0$ $\Rightarrow c_1 = 0$.

$\Rightarrow x = \omega\cos\theta \cdot g \cdot \frac{t^3}{3} + c_2$ } At $t=0$, $x=0$ $\Rightarrow c_2 = 0$.

Now $H = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2H}{g}}$.

$$\Rightarrow \boxed{x = \frac{\omega\cos\theta}{3}\sqrt{\frac{8H^3}{g}}\;\hat{i}}$$

x would be the deflection on $\hat{i}$ i.e. on east-side.

(Please do not write anything except the question number in this space)
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उम्मीदवारों को इस हाशिए में नहीं लिखना चाहिए
Candidates must not write on this margin

* Coriolis force is the reason why cyclones don't occur near equator
* Centrifugal force is which throws body off radially. Eg:- Rotating stone tied via string
* Euler's force causes tangential motion. Eg:- Child sliding back on horse when merry go round is accelerating.

13/20

Q2(b) Rutherford scattering is scattering of α-particle by a massive nucleus like gold.

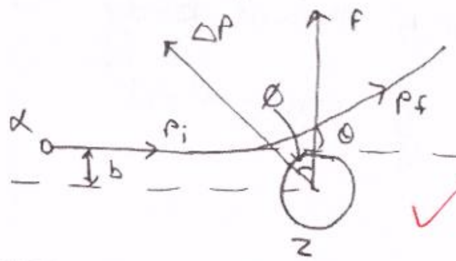

Fig 1. Rutherford scattering

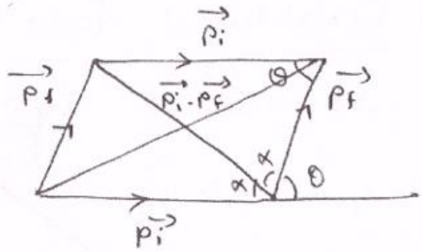

Fig 2. Momentum conservation

> Consider α-particle incident on gold nucleus.

> As gold nucleus >> α-particle, $|p_i| = |p_f|$

$\Rightarrow$ From figure 2, $\dfrac{|p_i - p_f|}{\sin\theta} = \dfrac{|p_i|}{\sin\alpha}$. Now as $|\vec{p_i}| = |\vec{p_f}|$, $\angle$ b/w $\vec{p_i}$ & $\vec{p_f}$ $= 2\alpha = \pi - \theta$.

$\Rightarrow |\vec{p_i} - \vec{p_f}| = 2p_i \sin\theta/2$ — ①.

Also, $\Delta P = \int F\cos\phi \, dt$. For coulombic force $F = k/r^2$ where $k = \dfrac{z_1 z_2 e^2}{4\pi\varepsilon_0}$.

> As coulombic force is central force, angular momentum is constant

$\Longrightarrow mr^2 \dfrac{d\theta}{dt} = J$

$\Rightarrow dt = \frac{mr^2}{J} d\phi$. } As J is constant, J = $p_i b$ initially.

$$\Rightarrow \Delta P = \int_{-\left(\frac{\pi-\theta}{2}\right)}^{(\pi-\theta)/2} \frac{k}{r^2} \cdot \cos\phi \cdot \frac{mr^2}{p_i b} \cdot d\phi = 2p_i \sin\theta/2$$

From eq. (1)

On solving, we get, $b = \frac{k}{2E_\alpha} \cot\frac{\theta}{2}$ } $E_\alpha = \frac{p_i^2}{2m}$

We know scattering cross-section $\sigma(\theta)$

$$= \frac{\text{\# particles scattered}}{\text{area-time} \times \text{Incident intensity}} = \frac{-b}{\sin\theta} \frac{db}{d\theta}.$$

$$\Rightarrow \sigma(\theta) = \frac{-k}{2E_\alpha} \cot\frac{\theta}{2} \times \frac{1}{\sin\theta} \cdot \frac{k}{2E_\alpha} - \operatorname{cosec}^2\theta/2 \cdot \frac{1}{2}.$$

$$\Rightarrow \boxed{\sigma(\theta) = \frac{k^2}{16E_\alpha^2} \operatorname{cosec}^4\left(\frac{\theta}{2}\right)}$$

Total cross-section $\sigma = \int_0^\pi \sigma(\theta)\, 2\pi \sin\theta\, d\theta$

$$\Rightarrow \sigma = \int_0^\pi \frac{k^2}{16E_\alpha^2} \operatorname{cosec}^4\left(\frac{\theta}{2}\right) 2\pi \sin\theta\, d\theta = \infty$$

Hence total cross-section turns out to be ∞.

Interpretations

(1) As impact parameter b increases, θ decreases rapidly $\Rightarrow$ Most of space in atom is empty.

(2) Total cross-section $= \infty \Rightarrow$ Scattering can occur for any value of b $\Rightarrow$ Range of coulombic force is ∞.

> However, practically it is limited due to shielding effect of electrons.

If $b = 0$, this experiment is used to find nucleus size & is estimated to be ~ 27 fm.

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Q2 (c)

Materials within Hooke's limit follow Hooke's law which states stress is $\propto$ strain.

$\therefore$ Young's Modulus $Y = \frac{P}{\Delta l / L} \Rightarrow Y = \frac{FL}{A \Delta l}$ — (1)

$\Rightarrow F = \left(\frac{YA}{L}\right) \Delta l$ { A = cross-section area, F = longitudnal force.

As it is elastic force, $F \propto kx$. We know in elastic force, $W = \frac{1}{2} k x^2$

Here $k = \frac{YA}{L}$; $x = \Delta l$

$\Rightarrow W = \frac{1}{2} \times \frac{YA}{L} \times \Delta l \times \Delta l = \frac{1}{2} (YA \Delta L) \times \text{strain}$

Now $YA\Delta L = FL$ from eq. (1) $\Rightarrow YA\Delta L = P(A)(L)$
$= P(\text{Volume})$

$\Rightarrow \frac{W}{\text{Volume}} = \frac{1}{2} \times P \times \text{strain}$ } $\Rightarrow$ Work done per-unit Volume $= \frac{1}{2}(\text{stress}) \times \text{strain}$

Numerical: $k = \frac{YA}{L} = \frac{2 \times 10^{11} \times 1 \times 10^{-6}}{2} = 10^5$ N/m

$\Rightarrow W = \frac{1}{2} k x^2 = \frac{1}{2} \times 10^5 \times (0.1)^2 \times 10^{-6} = \boxed{5 \times 10^{-4} J}$

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Work done per unit volume gives Proof Resilience i.e. maximum stress that wire can take

Q.3(a) Moment of inertia is the angular resistance offered by body to torque applied

We know, angular momentum $\vec{J} = \sum m_i [\vec{r_i} \times \vec{v_i}]$

$\Rightarrow \vec{J} = \sum m_i [\vec{r_i} \times (\vec{\omega} \times \vec{r_i})]$. Taking components, we get,

$$\vec{J} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \Rightarrow \vec{J} = \overleftrightarrow{I} \vec{\omega}.$$

where $I_{xx} = \sum m_i (y_i^2 + z_i^2)$, $I_{yy} = \sum m_i (x_i^2 + z_i^2)$

$I_{zz} = \sum m_i (x_i^2 + y_i^2)$; $I_{xy} = -\sum m_i x_i y_i$,

$I_{xz} = -\sum m_i x_i z_i$;

Where I_{xx}, I_{yy}, I_{zz} = principal moments of inertia.

$I_{xy}, I_{yz} \ldots$ = cross-products.

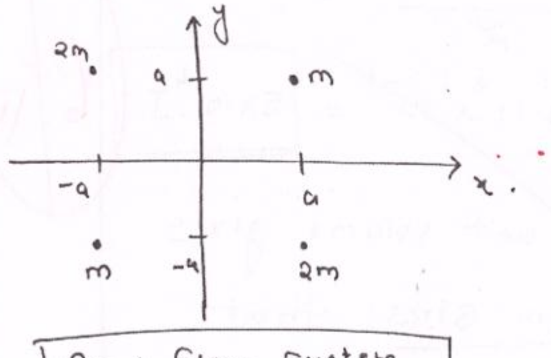

Fig 1. Given system

Hence, principal moments of inertia are,

$$I_{xx} = \sum m_i (y_i^2) = ma^2 + 2m(a^2) + 2m(a^2) + ma^2$$
$$= \underline{6ma^2}.$$

$$I_{yy} = \sum m_i (x_i^2) = ma^2 + 2m(a^2) + 2m(a^2) + ma^2$$
$$= \underline{6ma^2}.$$

$$I_{zz} = \sum m_i (x_i^2 + y_i^2) = m(2a^2) + 2m(2a^2) + 2m(2a^2) + m(2a^2) = \underline{12ma^2}$$

Hence $\boxed{I_{xx} = I_{yy} = 6ma^2}$ & $\boxed{I_{zz} = 12ma^2}$.

This also verifies the inertia addition theorem i.e. $\underline{I_x + I_y = I_z}$.

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$I_{xy} = I_{yx} = 2ma$
$I_{xz} = I_{zx} = 0$
$I_{yz} = I_{zy} = 0$

$|\vec{I} - \lambda I| = 0$
$4ma^2$
$8ma^2$
$12ma^2$
) is Ans.

Q.3 (b)

Central force is a force which is solely dependent on the position of particle.

For any force, we know,

$F_{(r,\theta)} = m\left[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}\right]$.

For central force, $(2\dot{r}\dot{\theta} + r\ddot{\theta})m = 0$

$\Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) = 0 \Rightarrow mr^2\dot{\theta} = J$ = constant

i.e. angular momentum is constant in central force.

(i) We know, $\vec{J} = \vec{r} \times \vec{p}$. As $\vec{J}$ is constant, its direction is constant & always $\perp$ to both $\vec{r}$ & $\vec{p}$. i.e. $\vec{r}$ is always in a plane $\perp$ to $\vec{J}$

$\Rightarrow \vec{r}$ is planar. Hence $\frac{d\vec{r}}{dt}$ is planar

$\Rightarrow$ Particle moves in a plane.

(ii) Areal velocity is ~~vel~~ area swept by particle per unit time.

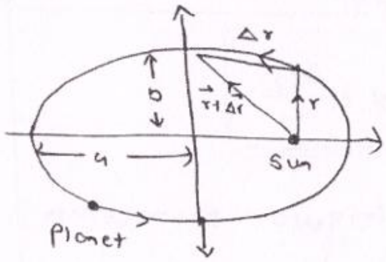

Fig 1. Example of central Force

Consider a central force like gravity. Areal velocity

$$V_{Ar} = \frac{dA}{dt} = \frac{d}{dt}\left(\frac{1}{2}\vec{r}\times(\vec{r}+\vec{\Delta r})\right)$$

$$\Rightarrow V_{Ar} = \frac{1}{2}\vec{r}\times\frac{d\vec{r}}{dt} \quad \left\{ \text{As } \vec{r}\times\vec{r} = 0 \right.$$

$$\Rightarrow V_{Ar} = \frac{1}{2}\vec{r}\times\vec{v} = \frac{1}{2}\frac{\vec{r}\times\vec{p}}{m} = \frac{J}{2m}.$$

Now as J is constant, V_{Ar} is constant.

(iii) Time period is time taken to complete one loop. $\Rightarrow T = \frac{\text{Area}}{\text{Areal velocity}} = \frac{\pi ab}{J/2m}$.

For planetary motion, $e = \sqrt{1-\frac{b^2}{a^2}}$ } e : eccentricity of ellipse.

~~$\Rightarrow b^2 = a^2(1-e^2) \Rightarrow b = a\sqrt{1-e^2}$.~~

~~$\Rightarrow T = \pi a$~~

Comparing with eq. of motion of particle,

$$\frac{J^2/km}{r} = 1+\sqrt{1+\frac{2EJ^2}{mk^2}}\cos\theta \equiv \frac{l}{r} = (1+e\cos\theta)$$

Eq. of conic

$$\Rightarrow l = \frac{b^2}{a} = J^2/km \Rightarrow b = \frac{J}{\sqrt{km}}a^{1/2}$$

$9\frac{1}{2}/15$

$\Rightarrow T = \frac{\pi}{J/2m} \cdot a \cdot \frac{J}{\sqrt{km}} a^{1/2} \Rightarrow \boxed{T \propto a^{3/2}}$

Here k = force constant, J = angular momentum.

∴ Time period for planetary motion depends only on major axis.

Examples of central forces include gravitational force, coulombic force, etc.

Q.3 (c)

Poiseuille's equation gives the velocity profile of a liquid : (1) Flowing through horizontal pipe (2) Is viscous (3) Is laminar.

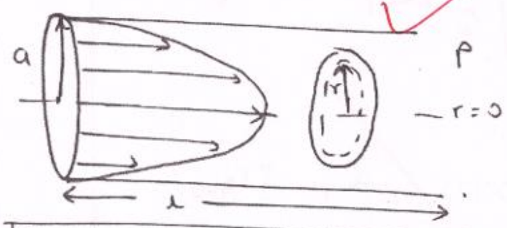

Fig 1. Velocity profile

We know, frictional force $= \eta A \frac{dv}{dz}$

For flow to happen, Frictional force = Pressure Force

$\Rightarrow \eta A \frac{dv}{dr} . 2\pi r l = P\pi r^2$.

Boundary condition, $V_{(r=a)} = 0$, we get,

$V = \frac{P}{4\eta L}(a^2 - r^2)$. Maximum flow will occur at $r = 0$

$\Rightarrow$ Max. velocity $V_{max} = \frac{Pa^2}{4\eta L}$.

Given $P = h\rho g = 0.15 \times 1000 \times 9.8 = 1470 \, N/m^2$

$\eta = 9.8 \times 10^{-3}$ poise $= 9.8 \times 10^{-4}$ SI units.

$\Rightarrow V_{max} = \frac{1470 \times 10^{-6}}{4 \times 9.8 \times 10^{-4} \times 0.4} = 0.9375 \, m/sec$

Now, $Re = \frac{\rho v D}{\eta}$ where $D = 2mm$.

$\Rightarrow Re = \frac{10^3 \times 0.9375 \times 2 \times 10^{-3}}{9.8 \times 10^{-4}}$ } Re = Reynold's number

$V = \Delta V = \frac{V}{2}$

$\Rightarrow$ $\boxed{Re = 1913.3}$

For Reynold's number < 2000, the flow is considered as laminar.

Limitations of Poiseuille's equation include – can be used only for horizontal pipe, without any external force.

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Q4. (a)

Stress (P) in material causes deformation in length, volume & orientation giving us three constants :-

Young's Modulus : $Y = \frac{P}{\Delta l/L}$ } P = stress, $\Delta l/L$ = Longitudnal strain

Bulk's Modulus : $k = \frac{P}{\Delta v/V}$ } $\Delta v/V$ = volume strain

Coefficient of Rigidity : $\eta = \frac{P}{\theta}$ } θ = shear.

σ = Poisson's ration = $\frac{\text{Lateral strain}}{\text{Longitudnal strain}}$.

(i) $Y = 3k(1-2\sigma)$

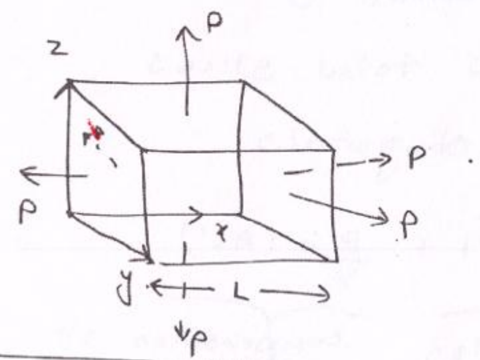


Fig1. Stress on cube

> Consider a cube of length L being pulled from all sides by stress P.

> In x direction,

$$L' = L + \frac{PL}{Y} - \frac{\sigma PL}{Y} - \frac{\sigma PL}{Y}$$

Extension due to stress in x-direction; Contraction due to stress in y & z-direction.

$\Rightarrow L' = L\left[1 + \frac{P}{Y}[1-2\sigma]\right]$

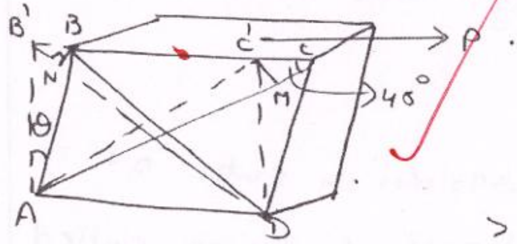
$\Rightarrow V' = (L')^3 \Rightarrow V' = L^3\left[1 + \frac{P}{Y}[1-2\sigma]\right]^3$

$\Rightarrow V' = V\left[1 + \frac{3P}{Y}[1-2\sigma]\right]$ Expanding binomially

$\Rightarrow \frac{\Delta V}{V} = \frac{3P}{Y}[1-2\sigma] \Rightarrow Y = \frac{3P}{\Delta V/V}[1-2\sigma]$

$\Rightarrow \boxed{Y = 3K(1-2\sigma)}$.

(ii) $Y = 2\eta(1+\sigma)$



> consider a stress P as shown creating shear θ.

> Now total stress

$= \frac{CM}{AC'} + \frac{BN}{DB'}$ } change in diagonals.

Total extension $= \underbrace{\frac{P}{Y}(AC')}_{\text{Extension}} + \underbrace{\frac{P}{Y}\sigma(AC')}_{\text{compression on other side}}$.

$\Rightarrow \frac{P}{Y}(AC')(1+\sigma) = CM$ ~~+ BN~~.

$\Rightarrow \frac{P}{Y}(1+\sigma) = \frac{CM}{AC'}$ ~~+ BN/AC~~ $= \frac{L\theta\cos 45^\circ}{AC'}$ } L = DC

$\Rightarrow \frac{P}{Y}(1+\sigma) = \frac{DC}{AC'} \cdot \frac{\theta}{\sqrt{2}} = \frac{\theta}{2} = \frac{P}{2\eta}$

$\Rightarrow \boxed{Y = 2\eta(1+\sigma)}$

(iii) From (i) & (ii),

$$\frac{Y}{2\eta} - 1 = \left(\frac{-Y}{3K} + 1\right)\frac{1}{2} = \sigma.$$

$$\Rightarrow \frac{Y}{2\eta} - 1 = \frac{-Y}{6K} + \frac{1}{2} \Rightarrow Y\left[\frac{1}{2\eta} + \frac{1}{6K}\right] = \frac{3}{2}.$$

$$\Rightarrow \boxed{Y = \frac{9\eta K}{3K+\eta}}$$

These relations help find the relative deformities if one is known.

9½/15

Q4 (b)

Euler's equation of motions give a most general equation for 3-D body. Such that,

$N_1 = I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3$

$N_2 = I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1$

$N_3 = I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2$

N = torque in body set

I_1, I_2, I_3 are principal moment of inertia.

ω = angular velocity.

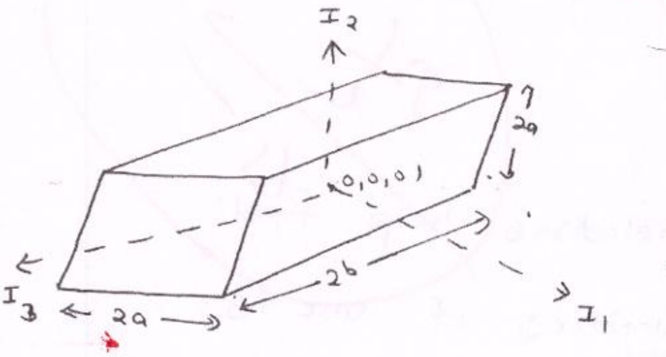


Fig 1. Parallelopiped

> For given parallelopiped, consider, I_1, I_2, I_3 to be principle axis. As 2 sides are equal, $I_1 = I_2$ by symmetry.

⟹ From Euler's equations, we have,

$N_1 = N_2 = N_3 = 0$ } As no external force acts on body.

$\Rightarrow N_3 = 0 = I_3\dot{w}_3 - (0)w_1w_2$ } As $I_1 = I_2$.

$\Rightarrow I_3\dot{w}_3 = 0 \Rightarrow$ $\boxed{w_3 = \text{constant}}$.

Now, $I_1\dot{w}_1 - (I_2 - I_3)w_3w_2 = 0$.

$\Rightarrow \dot{w}_1 = \left[\frac{(I_1 - I_3)w_3}{I_1}\right]w_2 = \Omega^2 w_2$ — (1)

Similiarly, $\dot{w}_2 = \Omega^2 w_1$ — (2)

$\Rightarrow \ddot{w}_1 = \Omega^2 w_1$ & $\ddot{w}_2 = -\Omega^2 w_2$ } from (1) & (2)

$\Rightarrow w_1 = A\sin\Omega t$ & $w_2 = A\cos\Omega t$.

4 | 20
0.5

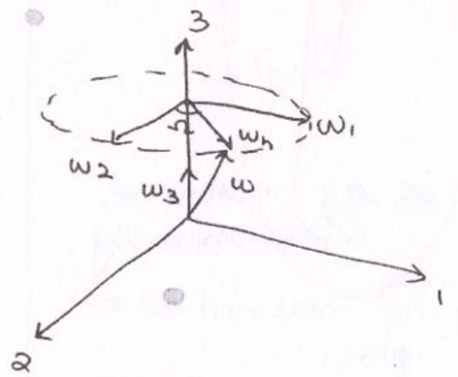


Fig 2. Precession.

$\Rightarrow$ Angular velocity given parallelopiped along $\hat{e}_3 = w_3$ is constant while along other 2 axis $\hat{e}_1$ & $\hat{e}_2$ is periodic with period $\boxed{t = \frac{2\pi}{\Omega}}$ i.e. axis of rotation rotates with $\Omega \Rightarrow$ Precession

This phenomenon is seen in top when it loses velocity.

Q.4(c) Invariance refers to retaining of physical form across inertial frames.

Volume element $dV = dx\,dy\,dz\,dt$ in relativistic regime.

We know $x' = \alpha(x - vt)$ $y' = y$ } Lorentz transform.
$t' = \alpha(t - xv/c^2)$ $z' = z$

where S' = coordinates in frame travelling with v along x-axis.

Fig 1. Inertial frames

$\Rightarrow dV' = dx'\,dy'\,dz'\,dt'$.

Now ~~$dx' = \alpha(dx - vdt)$~~ } ~~As dx measured simultaneously~~.

$dt' = \alpha(dt - dx\,v/c^2)$ } As measured at one place.

~~$\Rightarrow dx'/dy'/dz'/dt' =$~~

$dx = \alpha(dx' - vdt')$ } $dt' = 0$ as measured simultaneously in body's own frame.

$\Rightarrow dx' = dx/\alpha$

$\Rightarrow dV' = dx'dy'dz'dt' = \frac{dx}{\alpha}dy\,dz\,dt\,\alpha = \boxed{dV}$.

Hence volume element is invariant under Lorentz transformation.

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Q5 (a)

Einstein's Special Theory of Relativity gave rise to relativistic momentum & energy.

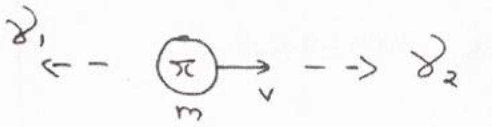

Fig 1. Pion - decay.

Given $p_i = mv = \frac{3}{4}mc$

We know, relativistic energy $E = \sqrt{p^2c^2 + m_o^2c^4}$

$\Rightarrow E_i = \sqrt{\frac{9}{16}m^2c^4 + m^2c^4} = \frac{5}{4}mc^2$ — (1).

Energy of photon $= pc \Rightarrow E_i = p_1c + p_2c$ — (2)

Conserving momentum,

$\vec{p_i} = \vec{p_2} - \vec{p_1} \Rightarrow p_2 - p_1 = \frac{3}{4}mc$ — (3).

From (2), $\frac{5}{4}mc = p_1 + p_2$ — (4). From eq. (3) & (4),

$2p_2 = 2mc \Rightarrow p_2 = mc \Rightarrow p_1 = \frac{mc}{4}$

$\therefore$ Relativisitic energies are :-

$E_2 = mc^2$ — photon emitted in π's direction

$E_1 = \frac{mc^2}{4}$ — photon emitted in opposite direction

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This decay happens through electromagnetic interaction

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Q5 (b).

Earth rotates around sun due to gravitational force which is conservative force $\Rightarrow$ Mechanical energy remains constant.

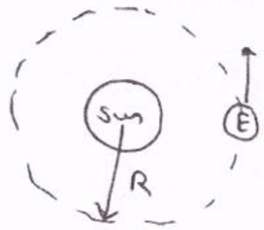

Fig 1. Earth Rotation

We know for conservative force,

$$\frac{J^2/km}{r} = 1 + \sqrt{1 + \frac{2EJ^2}{mk^2}} \cos\theta .$$

where $k = GMm$ for gravitation.

If the orbit is assumed circular, eccentricity

$$e = 0 \Rightarrow \sqrt{1+\frac{2EJ^2}{mk^2}} = 0 \Rightarrow E = -\frac{mk^2}{2J^2} \quad -(1)$$

We know, orbital velocity $= \sqrt{\frac{GM}{R}}$.

$$\Rightarrow E = PE + KE = -\frac{GMm}{R} + \frac{1}{2}\frac{GMm}{R} = -\frac{GMm}{2R}.$$

Now if velocity increased by 50%, $v = \frac{3}{2}\sqrt{\frac{GM}{R}}$

$$\Rightarrow E = -\frac{GMm}{R} + \frac{9}{4}\left(\frac{1}{2}\right)\frac{GMm}{R} \approx -\frac{GMm}{R}\left[-1+\frac{9}{8}\right] \approx \underline{0}.$$

i.e. total $E = 0 \Rightarrow$ **$e = 1 \Rightarrow$ Parabola**

Actual value is increase by $\sqrt{2}$ times i.e. $\sim 41\%$.

Not Correct

4/10

Qs (c).

Earth can be assumed to be a sphere of uniform density.

> we know for sphere $I = 2/5\, MR^2$

⇒ Angular momentum $J = I\omega$.

$I = \frac{2}{5} \times 6\times10^{24} \times (6.4\times10^{6})^2 = 9.83\times10^{37}\ kg\,m^2$.

Earth rotates 2π radians in 24 hrs

⇒ $\omega = \frac{2\pi}{24\times3600} = 7.27\times10^{-5}$ rad/sec

⇒ Angular momentum = $J = 7.148\times10^{33}\ kg\frac{m^2}{s}$.

Rotational KE $= \frac{1}{2} I\omega^2 = 2.6\times10^{29}$ J.

Given: energy used per second $= 3.5\times10^{9}\times10^{3}$
$= 3.5\times10^{12}$ J

⇒ Time to exhaust energy $= \frac{2.6}{3.5}\times10^{17}$

⇒ $T = 7.42\times10^{16}$ secs

∴ Time to exhaust earth's rotational KE will be ~ 235 Cr years

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Q5 (d)

Torque is the moment of force about a particular axis.

We know, $KE = \sum \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i (\vec{v_i} \cdot \vec{v_i})$

$\Rightarrow KE = \sum \frac{1}{2} m_i \left[(\vec{\omega} \times \vec{r_i}) \cdot (\vec{\omega} \times \vec{r_i}) \right]$

Expanding, $KE = \sum \frac{1}{2} m_i r_i^2 \omega^2$.

We know I = moment of inertia = $\sum \frac{1}{2} m_i r_i^2$.

$\Rightarrow$ KE for rotation $= \frac{1}{2} I \omega^2$.

$\Rightarrow T = \frac{1}{2} I \omega^2 \Rightarrow \frac{dT}{dt} = I \omega \cdot \frac{d\omega}{dt}$.

We know ~~$I\, d\omega/dt$~~ Torque $= G = I\alpha$; α = angular acceleration.

$$\Rightarrow \frac{dT}{dt} = \vec{G} \cdot \vec{\omega}$$

Hence, change of rotational KE is dot-product of torque & angular velocity.

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Answer Questions in NOT MORE THAN the Word Limit specified for each in the Parenthesis.
(Specimen Answer Booklet - For Practice Purpose Only)

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Candidates must not write on this margin

Q5 (c)

When the energy of particle is more than rest mass energy, relativistic effects take precedence.

Given :- $p = 1 MeV/c \Rightarrow pc = 1 MeV$.

We know ; $E = \sqrt{p^2c^2 + (m_0c^2)^2}$ $\Big\}$ $m_0c^2 = 0.51 MeV$ for e^-.

$\Rightarrow E = \sqrt{1 + (0.51)^2} = 1.122 MeV$.

Also $E = KE + m_0c^2 \Rightarrow KE = 1.122 - 0.51$

$\Rightarrow$ $KE = 0.612 MeV$.

Also, $E = mc^2 = \frac{m_0}{\sqrt{1 - v^2/c^2}} \cdot c^2$

$\Rightarrow \sqrt{1 - \frac{v^2}{c^2}} = \frac{0.51}{1.122} \Rightarrow \frac{v}{c} = 0.89$.

$\Rightarrow$ velocity = $0.89c = 2.67 \times 10^8$ m/sec.

· In relativistic regime, apparent mass of object is always more than rest mass.

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